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AIAA 81-4128

Airborne Method to Minimize Fuel with Fixed Time-of-Arrival Constraints

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Introduction

THE world's transportation system is being adversely affected by the decreasing supply of petroleum coupled with the increasing demand for travel and shipment of cargo. This situation especially affects the air transportation industry where there is no current alternative to jet fuel. Thus, it has become mandatory to seek new ways to conserve fuel by building more energy-efficient aircraft and by flying existing aircraft along more fuel-efficient flight paths.

One concept of near-term impact is application of trajectory optimization principles to generate the vertical profile (and the inherent elevator/throttle control sequence) to minimize fuel and other costs in flying over a fixed horizontal path (between a fixed city pair). This Note focuses on the process for meeting destination time-of-arrival constraints and the potential benefits that an on-board flight management system having this capability might have.

It is assumed that 1) aircraft will soon be nominally flown along minimum-fuel vertical flight paths, 2) increasing delays of variable length will occur at the arrival point because of expected congestion, and 3) the air traffic control (ATC) system can anticipate this congestion and inform the pilot what his delay will be before he arrives in the terminal area (i.e., the controller assigns the pilot an expected time-of-arrival). The pilot has a choice of two strategies to follow: 1) continue to fly his nominal minimum fuel path and then go into a minimum-fuel-rate holding pattern to absorb the delay at the end of the cruise segment; or 2) slow down so that he arrives at the terminal area within an acceptable tolerance of the assigned time-of-arrival.

The algorithm developed in this study generates the vertical flight path between a city pair which minimizes fuel and meets the delayed time-of-arrival constraint of option 2 above. The fuel reduction of using this strategy is compared to that of option 1 as a function of delay time. Further detail can be found in Ref. 1.

Received October 17, 1980.

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Previous significant work in generating optimum vertical flight paths that minimize fuel or direct operating costs are summarized in Ref. 2. Schultz and Zagalsky developed an algorithm that solved the minimum-fuel, fixed-range, fixed-time problem.[3] Erzberger and his colleagues extended this algorithm to minimize direct operating costs in the presence of winds.[4] They also developed efficient computer programs for generating typical flight profiles.[5] This study represents an extension of Erzberger's work to include control of time-of-arrival.

Approach

The longitudinal point mass model of the aircraft can be described with five state variables: airspeed V, flight-path angle γ, altitude h, horizontal range x, and mass m (Ref. 4). For optimization purposes, it is adequate to combine h and V into the specific energy state

$$E = h + V^2/2g \tag{1}$$

with its time derivative

$$\dot{E} = V(T-D)/mg \tag{2}$$

The rate of change of γ is rapid, relative to the other state variables, so that it can be considered as a control. Also, the mass burn rate can be ignored in optimizing the flight path. This leaves two state equations for range and energy.

The cost function to be minimized is of the form

$$J = \int_{t_0}^{t_f} (K_f \dot{f} + K_t)\,\mathrm{d}t \tag{3}$$

where K_f and K_t are the unit costs of fuel (\$/lb) and time (\$/h), and $\dot{f}$ is the fuel flow rate. The problem then is to choose the sequence of controls that satisfy the constraints and minimize the cost of flight governed by Eq. (3).

Erzberger uses the assumption that aircraft energy monotonically increases during climb and monotonically decreases during descent.[4] Energy E is then used as the independent variable by dividing Eq. (2) into Eq. (3). This result produces a constant value for the adjoint variable λ on an optimum trajectory. This variable is the cost of flight during cruise expressed as

$$\lambda = (K_f \dot{f} + K_t)/(V + V_w) \tag{4}$$

where V_w is the longitudinal component of windspeed. The Hamiltonian that is minimized during climb and descent is

$$\min_{\pi, V} \frac{K_f \dot{f} + K_t - \lambda(V+V_w)}{(T-D)V/mg} \tag{5}$$

where controls are throttle setting π and airspeed. Equation (5) is incorporated into the flight management system as the algorithm for on-board generation of the near-optimum flight path.

If the cost of time K_t in Eqs. (4) and (5) is set to zero, the result is the minimum fuel flight path. If this produces the nominal time-of-arrival, then to realize arrival delays beyond this point requires that the coefficient K_t be set negative. The minimum value of λ occurs where cruise airspeed produces minimum fuel rate $\dot{f}_{\min}$. This is found by setting the derivative of Eq. (4) with respect to V to zero and by using the fact that $\dot{f}$ is minimum where $\partial \dot{f}/\mathrm{d}V$ is zero. This condition yields

$$K_{t_{\min}} = -K_f \dot{f}_{\min} \tag{6}$$

Using these values of (K_f, K_t) and λ of zero in Eq. (5) produces the maximum practical delay time that can be achieved by reduced flight speed. Further flight delay should

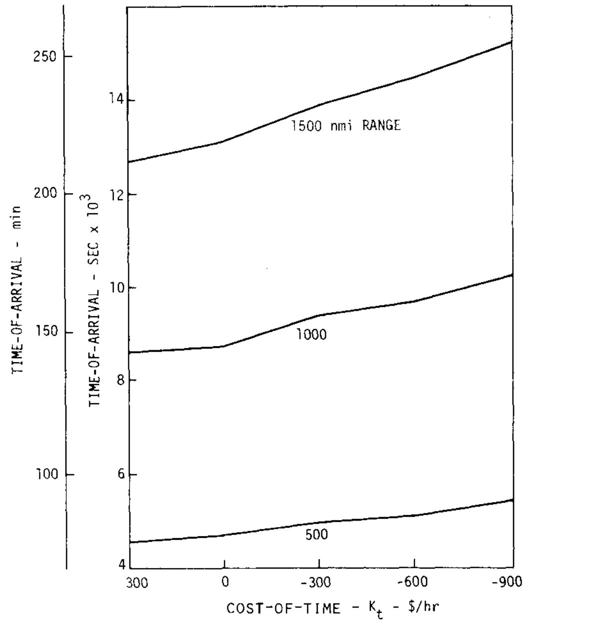

Fig. 1 Time-of-arrival variations as a function of range and cost of time K_t; initial weight of 68,182 kg (150,000 lb) and cruise altitude fixed at 10 km (33,000 ft).

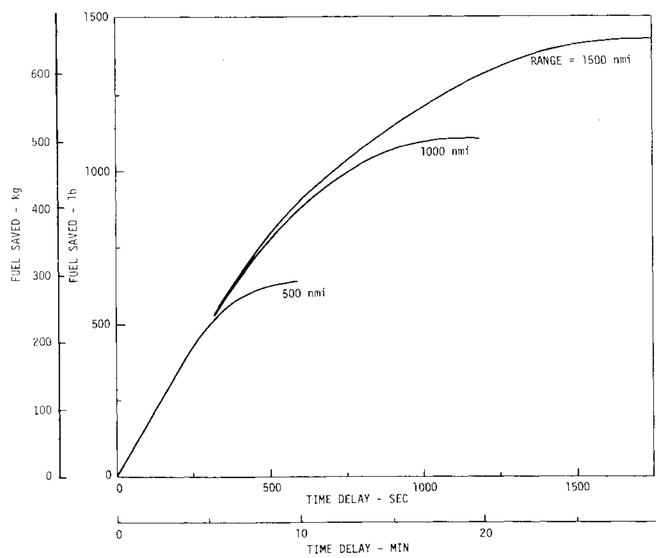

Fig. 2 Fuel saved using option 2 as a function of time delay and range.

be absorbed by path stretching in the holding pattern with airspeed set to achieve $\dot{f}_{min}$.

Results

A computer program has been devised to generate vertical flight profiles using an algorithm that minimizes Eq. (5) during climb and descent and iteratively selects the value of K_t that produces the desired time-of-arrival. This algorithm is suitable for on-board mechanization.

Figure 1 shows the time delays that can be achieved for a medium-range tri-jet aircraft weighing 150,000 lb at takeoff as a function of cost-of-time K_t and total range traveled. Cost of fuel is fixed at $0.15/lb. Thus, for example, a time delay of 25 min can be achieved on a 1000 n.mi. route by setting the cost-of-time at −$900/h.

A benefit of having an on-board algorithm to control time-of-arrival in terms of saving fuel was determined by comparing the fuel used by such trajectories with that used by following the option 1 strategy described earlier. Figures 2 and 3 illustrate the amount and percent of fuel that can be saved on flights with up to 1750 s of time delay for the 150,000 lb aircraft. Figure 2 shows that for a 1500 n.mi. trip with 1750 s of delay, about 1450 lb of fuel can be potentially saved. Approximately 500 lb of fuel can be saved for a 5-min delay, independent of range.

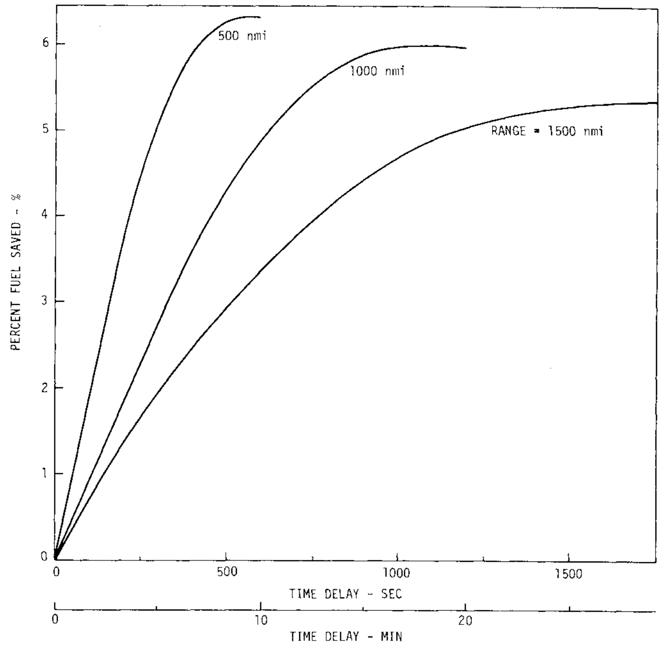

Fig. 3 Percentage fuel saved using option 2 as a function of time delay and range.

Figure 3 indicates the percentage of fuel saved for the same cases shown in Fig. 2. Up to 6% of the fuel can be saved with this capability. These values are computed by dividing the reduced fuel amount by that used for controlled time-of-arrival.

Conclusion

This paper presents a method for generating a minimum-fuel, fixed-range, fixed-time-of-arrival flight path in an on-board flight management system computer. Such a system has the potential of saving up to 6% of the fuel that would be consumed in time-of-arrival delays of up to 30 min for a medium-range tri-jet transport aircraft.

Acknowledgments

This work has been supported by NASA Langley Research Center under Contract No. NAS1-15497.

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